

The University of Chicago

A PROJECTIVE GENERALIZATION
OF EUCLIDEAN PARALLELISM
OF SURFACES

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1933

By

MARION LEIGH MacQUEEN

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I. INTRODUCTION

The purpose of this paper is to develop a projective analogue of the concept of Euclidean parallelism of surfaces. In order to facilitate the explanation of this idea, let us recall the definition and some of the properties of parallel surfaces in Metric Differential Geometry. If we consider a surface S in ordinary metric space M_3 , and lay off segments of constant length upon its normals, these segments being measured from the surface, the locus of their other end points is a surface $\bar{S}$. The surfaces S and $\bar{S}$ are said to be parallel¹ in the metric sense. Two such parallel surfaces then have the property that they possess the same normal at corresponding points, and also the property that the distance measured along the normal between corresponding surface points is constant, i.e., is the same for all pairs of corresponding points. The developables of the common normal congruence cut both surfaces in their lines of curvature, and the tangents of corresponding lines of curvature at corresponding points are parallel. The tangent planes at two corresponding surface points are parallel and therefore intersect in a line which lies in the plane at infinity. It may also be remarked that there are infinitely many surfaces parallel to a given surface.

In order to formulate a projective generalization of metric parallelism of surfaces we begin by replacing the metric normal congruence by the projective normal congruence, and so

¹Eisenhart, Differential Geometry, Ginn and Co., 1909, p. 178.

consider two surfaces in ordinary projective space S_3 with a common projective normal congruence. The developables of this congruence intersect both surfaces in the projective lines of curvature, which form conjugate nets. We then demand, in analogy to the metric parallelism of the tangent planes, that the tangent planes at corresponding points of the two surfaces intersect on a fixed plane. Two surfaces so related are said to be parallel in the projective sense.

For the basis of our study of projective parallelism we employ one of the well-known transformations of surfaces, namely, the fundamental transformation. Two surfaces are said to be in the relation of a fundamental transformation, or transformation F , in case their points are in a one-to-one correspondence such that the lines joining corresponding points form a congruence whose developables intersect both surfaces in conjugate nets, neither surface being a focal surface of the congruence. The congruence is called the conjugate congruence of the transformation, because it is conjugate to both nets. The tangent planes at corresponding points of the two surfaces intersect in the lines of the harmonic congruence of the transformation, which is harmonic to both nets. By choosing the projective normal congruence as the common conjugate congruence of the transformation F , we provide, as previously stated, a projective substitute for the metric normal congruence. Furthermore, we assume that the developables of the harmonic congruence are indeterminate, that is, that corresponding tangent planes of the two surfaces intersect in the lines of a fixed plane. This assumption affords us a projective substitute for the metric parallelism of the tangent planes. We are now able to reformulate our definition of projective parallelism

in the following way:

Two surfaces are said to be projectively parallel in case they are in the relation of a fundamental transformation with the projective normal congruence as the conjugate congruence and with the developables of the harmonic congruence indeterminate.

In Section II will be found the analytic basis of our study. We start with a rather general completely integrable system of partial differential equations, namely, the system for two surfaces in the general analytic one-to-one point correspondence in ordinary projective space S_3 , and a group of transformations that leaves this configuration invariant. We then modify this system of equations so that every pair of integral surfaces will be in the relation of a fundamental transformation.

In Section III we reduce the system of equations thus obtained to a canonical form, and employ it to obtain the conditions that the conjugate congruence of the transformation becomes the projective normal congruence. The parametric conjugate nets on both surfaces are therefore their projective lines of curvature. Furthermore, in Section IV, we impose the condition that the developables of the harmonic congruence are indeterminate, so that the tangent planes at corresponding points of the two surfaces intersect in the lines of a fixed plane. We conclude this section by formulating the definition of projective parallelism and a theorem describing the analytic machinery for studying it.

In Section V we investigate with reference to our projective configuration the metric property that a given surface possesses infinitely many parallel surfaces, and seek to determine whether a given surface admits of infinitely many projectively parallel surfaces. The conclusion is that it does not do so in general.

The assumption that the conjugate congruence of the transformation F is the congruence of projective normals is dropped in Section VI. We study in the familiar way the configuration composed of two surfaces in ordinary space in the relation of a fundamental transformation having a general conjugate congruence and with the developables of the harmonic congruence indeterminate. For the analytic basis of this study we employ a somewhat different canonical form of our system of differential equations. We are thus led to a characterization of surfaces which are projectively parallel in a modified sense. A given surface admits of infinitely many surfaces which are projectively parallel in this modified sense.

Finally, in Section VII, we return to the case where the conjugate congruence of the transformation F is the congruence of projective normals, and investigate certain properties of projective parallelism. For example, the projective analogue of the plane at infinity of metric parallelism is studied. Certain invariants of a net are calculated, and the change in them due to the special transformation F under consideration is computed. A conjugate net having certain properties is considered, and necessary and sufficient conditions that the transformation shall preserve these properties are found. A projective analogue of the metric relation between the points which are the Laplace transforms of points on a normal to a system of parallel surfaces and the focal points on the normal is determined. Employing a local coordinate system, we consider the axes and rays at corresponding points of projectively parallel surfaces. Finally, the correspondence between certain nets of curves on two projectively parallel surfaces is considered.

II. ANALYTIC BASIS. FUNDAMENTAL TRANSFORMATION.

In this section we place as fundamental the well-known system of differential equations frequently used in the study of the configuration composed of two surfaces S_x, S_y , in ordinary projective space S_3 , with their points in a one-to-one correspondence. We then modify this system of equations so that every pair of integral surfaces will be in the relation of a fundamental transformation.

Let

$$x^{(k)} = x^{(k)}(u, v), \quad y^{(k)} = y^{(k)}(u, v) \quad (k = 1, 2, 3, 4)$$

be the parametric equations of two surfaces S_x and S_y . These surfaces have their points in a one-to-one correspondence, such that corresponding points P_x, P_y have the same curvilinear coordinates u, v . If each point P_y does not lie in the tangent plane of S_x at the corresponding point P_x , then S_x, S_y are a pair of integral surfaces of a system of differential equations¹ of the form

$$\begin{aligned} (1) \quad x_{uu} &= px + \alpha x_u + (\beta x_v + Ly, \\ x_{uv} &= cx + ax_u + bx_v + My, \\ x_{vv} &= qx + \gamma x_u + \delta x_v + Ny, \\ y_u &= fx + mx_u + sx_v + Ay, \\ y_v &= gx + tx_u + nx_v + By. \end{aligned}$$

The coefficients of this system are connected by means of twelve integrability conditions. Of these conditions the last four,

¹Lane, Projective Differential Geometry of Curves and Surfaces, University of Chicago, 1932, p. 183 et seq.

obtained by equating the two expressions for y_{uv} that are consequences of the last two equations of (1), are

$$\begin{aligned}
 (2) \quad & t_u + t\alpha + an + mB + g = m_v + am + s\gamma + tA, \\
 & s_v + s\delta + bm + nA + f = n_u + bn + t\beta + sB, \\
 & g_u + pt + cn + fB = f_v + qs + cm + gA, \\
 & B_u + tL + nM = A_v + sN + mN.
 \end{aligned}$$

We shall now modify system (1) so that every pair of integral surfaces will be in relation F. For this purpose we shall determine the developables of the congruence of lines λ_{xy} joining corresponding points of the surfaces S_x, S_y . The arbitrary point P_z defined by

$$z = y + kx \quad (k \text{ scalar})$$

is on a line λ_{xy} . As λ_{xy} varies over a ruled surface of the family $dv - \lambda du = 0$, the point P_z generates a curve whose tangent at P_z is determined by P_z and the point P_z' given by

$$z' = y_u + y_v \lambda + k(x_u + x_v \lambda) + k'x.$$

Expressing z' as a linear combination of x, x_u, x_v, y and equating to zero the coefficients of x_u, x_v , we find the conditions on k, λ necessary and sufficient that λ_{xy} may generate a developable and have P_z for a focal point, namely

$$\begin{aligned}
 m + k + t\lambda &= 0, \\
 s + (n + k)\lambda &= 0.
 \end{aligned}$$

If we eliminate k and replace λ by dv/du we obtain the differential equation of the developables of the congruence of lines λ_{xy} ,

$$(3) \quad s \, du^2 - (m - n)du \, dv - t \, dv^2 = 0.$$

Eliminating λ , we obtain

$$(4) \quad k^2 + (m + n)k + mn - st = 0.$$

The two points z corresponding to the two roots of this equation are the focal points of λ'_{xy} .

If now $s = t = 0$, $m \neq n$, the developables of the congruence of lines xy are determinate and intersect S_x and S_y in the parametric curves thereon, the foci η, ζ of a line λ'_{xy} being given by

$$(5) \quad \eta = y - mx, \quad \zeta = y - nx.$$

We shall suppose from now on that $s = t = 0$, and in order that S_y may not be a focal surface of the congruence we shall suppose that $mn \neq 0$.

It is known that the parametric curves on S_x form a conjugate net in case $M = 0$. Let us suppose from now on that this condition is satisfied, and in order that S_x may not be developable we shall suppose that $LN \neq 0$. Under the suppositions $s = t = M = 0$, the integrability conditions (2) become

$$(6) \quad \begin{aligned} m_v + a(m - n) &= g + mB, \\ n_u + b(n - m) &= f + nA, \\ f_v + gA + c(m - n) &= g_u + fB, \\ A_v &= B_u. \end{aligned}$$

We are able to calculate the system of equations corresponding to (1) when the roles of x and y are interchanged. For the present the only coefficient that is needed is the one corresponding to M , which vanishes when the parametric net on S_y is a conjugate net. This condition, which we shall suppose from now on

to be satisfied, is found to be

$$(7) \quad fn(m_v + am) + gm(f + bm) - mn(f_v + cm) = 0.$$

By means of (6) this condition can be shown to be equivalent to

$$(8) \quad (f/m)_v = (g/n)_u$$

if $m - n \neq 0$.

The conditions which we have now imposed insure that every pair of integral surfaces of system (1) will be in relation F. Thus we reach the following conclusion:

Any system (1) such that every pair of integral surfaces is in the relation of a fundamental transformation, the surface S_x being non-developable, can be reduced to the form

$$(9) \quad \begin{aligned} x_{uu} &= px + \alpha x_u + \beta x_v + Ly, \\ x_{uv} &= cx + ax_u + bx_v, \\ x_{vv} &= qx + \gamma x_u + \delta x_v + Ny, \\ y_u &= fx + mx_u + Ay, \\ y_v &= gx + nx_v + By \end{aligned} \quad (mnLN \neq 0).$$

The resulting twelve integrability conditions for system (9) can now be written as follows:

$$(10) \quad \begin{aligned} a_u + ab + c &= \alpha_v + \beta\gamma, \\ b_v + ab + c &= \delta_u + \beta\gamma, \\ b_u + b^2 + a\beta &= \beta_v + b\alpha + \beta\delta + p + nL, \\ a_v + a^2 + b\gamma &= \gamma_u + a\delta + \gamma\alpha + q + mN, \\ c_u + bc + ap &= p_v + c\alpha + q\beta + gL, \\ c_v + ac + bq &= q_u + c\delta + p\gamma + fN, \\ aL &= L_v + BL + \beta N, \\ bN &= N_u + AN + \gamma L, \end{aligned}$$

$$\begin{aligned}
 (10) \quad & an + mB + g = m_v + am, \\
 & bm + nA + f = n_u + bn, \\
 & g_u + cn + fB = f_v + cm + gA, \\
 & A_v = B_u.
 \end{aligned}$$

From these conditions, G. M. Green, using a somewhat different notation, has shown¹ that

$$(11) \quad (a + \delta + B)_u = (b + \alpha + A)_v.$$

¹Green, "Memoir on the General Theory of Surfaces, etc.," Trans. Amer. Math. Soc., Vol. 20 (1919), p. 152.

III. CANONICAL FORM. PROJECTIVE NORMAL.

In this section we obtain and employ a canonical form of the system of equations (9) defining two surfaces S_x, S_y in relation F . We use this canonical form to write the conditions that the line joining corresponding points P_x, P_y of the surfaces S_x, S_y may be the projective normal of S_x at P_x . In this work we follow closely a method used by Lane in obtaining a canonical form of a system of equations used by Green for the surface S_x . By introducing the system of equations analogous to (9) but with the roles of x and y interchanged, we are able to find the analytic conditions characterizing the corresponding canonical form for the differential equations of S_y . Finally, we impose the condition that the line joining corresponding points P_x, P_y of the surfaces S_x, S_y may be the projective normal of S_y at P_y . Our configuration then consists of two surfaces S_x, S_y in the relation of a fundamental transformation, with the projective normal congruence as the common conjugate congruence of the transformation. The parametric conjugate nets on both surfaces are therefore the projective lines of curvature of S_x, S_y .

The projective differential geometry of a conjugate net on a curved surface in ordinary space was studied by Green,¹ who used a certain system of two partial differential equations of the second order. For the net N_x one of these equations, in a notation different from that used by Green, is the Laplace equation appearing as the second of (9). The other is found by

¹Green, "Projective differential geometry of one-parameter families of space curves, etc.," Amer. Journ. Math., Vol. 37 (1915), p. 215, and Vol. 38 (1916), p. 287.

eliminating y from the first and third equations of (9) and solving the result for x_{uu} . Our equations corresponding to those of Green may be written in the form

$$(12) \quad \begin{aligned} x_{uu} &= a'x_{vv} + b'x_u + c'x_v + d'x, \\ x_{uv} &= ax_u + bx_v + cx, \end{aligned}$$

where

$$a' = L/N, \quad b' = \alpha - a'\gamma, \quad c' = \beta - a'\delta, \quad d' = p - a'q.$$

Besides the invariant coefficient L/N , other invariants for the net N_x used by Green are the Laplace-Darboux point invariants H , K , the Weingarten invariants $W^{(u)}$, $W^{(v)}$, the tangential invariants $\mathcal{H}, \mathcal{K}$, and the invariants $\mathcal{B}', \mathcal{C}', \mathcal{D}$, and R which are expressed in terms of the coefficients of system (9) by the formulas

$$(13) \quad \begin{aligned} H &= c + ab - s_u, & K &= c + ab - b_v, \\ W^{(u)} &= 2b_v + a_u - \delta_u - E_u - (\log L)_{uv}, \\ W^{(v)} &= 2a_u + b_v - \alpha_v - A_v - (\log N)_{uv}, \\ \mathcal{H} &= K + W^{(u)} = a_u + \beta\gamma - E_u - (\log L)_{uv} \\ &= (N/L)(\beta_u + \beta b - \beta A - \beta(\log L)_u), \\ \mathcal{K} &= H + W^{(v)} = b_v + \beta\gamma - A_v - (\log N)_{uv} \\ &= (L/N)(\gamma_v + \gamma a - \gamma B - \gamma(\log N)_v), \\ \mathcal{D} &= (L/N)(\gamma_u - 2\gamma b + \alpha\gamma + mN) - (\beta_v - 2\beta a + \beta\delta + nL), \\ 8\mathcal{B}' &= 4a + 2(N/L)\beta - 2\delta + (\log N/L)_v, \\ 8\mathcal{C}' &= 4b + 2(L/N)\gamma - 2\alpha + (\log L/N)_u, \\ R &= (L/N)\mathcal{B}'^2 + \mathcal{C}'^2. \end{aligned}$$

Lane has obtained¹ a canonical form of the equations used by Green in the study of a surface referred to a conjugate

¹Lane, "Contributions to the theory of conjugate nets," Amer. Journ. Math., Vol. 49, No. 4, 1927.

net. The choice of proportionality factor leading to this canonical form is precisely that which yields Fubini's normal coordinates. In order to obtain a canonical form of system (9) we shall follow closely the method used by Lane.

If we make the transformation

$$(14) \quad x = \lambda \bar{x}$$

on system (9), we obtain a system of the same form with the following coefficients:

$$(15) \quad \begin{aligned} \bar{p} &= p + \alpha \lambda_u / \lambda + \beta \lambda_v / \lambda - \lambda_{uu} / \lambda, & \bar{\delta} &= \delta - 2 \lambda_v / \lambda, \\ \bar{\alpha} &= \alpha - 2 \lambda_u / \lambda, & \bar{N} &= N / \lambda, \\ \bar{\beta} &= \beta, & \bar{F} &= \lambda(f + m \lambda_u / \lambda), \\ \bar{L} &= L / \lambda, & \bar{m} &= \lambda m, \\ \bar{c} &= c + a \lambda_u / \lambda + b \lambda_v / \lambda - \lambda_{uv} / \lambda, & \bar{A} &= A, \\ \bar{a} &= a - \lambda_v / \lambda, & \bar{g} &= \lambda(g + \\ & & n \lambda_v / \lambda), \\ \bar{b} &= b - \lambda_u / \lambda, & & \\ \bar{q} &= q + \delta \lambda_v / \lambda + \gamma \lambda_u / \lambda - \lambda_{vv} / \lambda, & \bar{n} &= \lambda n, \\ \bar{\gamma} &= \gamma, & \bar{B} &= B. \end{aligned}$$

The invariants (13) of N_x are absolutely unchanged by transformation (14).

Condition (11) shows that there exists a function q such that

$$(16) \quad \begin{aligned} q_u &= \alpha + b + A + (\log N)_u + 3(\log L/N)_u/2 - 2(\log R)_u, \\ q_v &= a + \delta + B + (\log L)_v + (\log L/N)_v/2 - 2(\log R)_v, \end{aligned}$$

Equations (15) show that the effect of the transformation (14) on these derivatives is given by

$$\bar{q}_u = q_u - 4 \lambda_u / \lambda, \quad \bar{q}_v = q_v - 4 \lambda_v / \lambda.$$

Therefore, if we choose

$$(17) \quad \lambda = e^{q/4},$$

we make $\bar{q}_u = \bar{q}_v = 0$. This leads us to the following statement:

The resulting differential equations (9) with coefficients (15) for the choice of λ defined by equation (17) constitute a canonical form of system (9). This canonical form is characterized by the conditions

$$(18) \quad \begin{aligned} \bar{\alpha} + \bar{\gamma} + \bar{A} + (\log N)_u + 3(\log L/N)_u/2 - 2(\log R)_u &= 0, \\ \bar{a} + \bar{\delta} + \bar{B} + (\log L)_v + (\log L/N)_v/2 - 2(\log R)_v &= 0. \end{aligned}$$

The transformation

$$(19) \quad \bar{u} = \varphi(u), \quad \bar{v} = \psi(v)$$

leaves the parametric net invariant, but changes the coefficients of system (9) according to the formulas

$$(20) \quad \begin{aligned} \bar{p} &= p/\varphi_u^2, & \bar{q} &= q/\psi_v^2, & \bar{n} &= n, \\ \bar{\alpha} &= (\alpha - \varphi_{uu}/\varphi)\varphi_u, & \bar{\gamma} &= \gamma\varphi_u/\psi_v^2, & \bar{B} &= B/\psi_v, \\ \bar{\beta} &= \beta\psi_v/\varphi_u^2, & \bar{\delta} &= (\delta - \psi_{vv}/\psi_v)/\psi_v, & \bar{L} &= L/\varphi_u^2, \\ \bar{c} &= c/\varphi_u\psi_v, & \bar{r} &= r/\varphi_u, & \bar{N} &= N/\psi_v^2, \\ \bar{a} &= a/\psi_v, & \bar{m} &= m, & \bar{A} &= A/\varphi_u, \\ \bar{b} &= b/\varphi_u, & \bar{g} &= g/\psi_v, \end{aligned}$$

The relative invariants (13) are changed according to the formulas

$$\begin{aligned} \bar{B}' &= \frac{1}{\psi_v} B', & \bar{R} &= \frac{1}{\varphi_u^2} R, \\ \bar{C}' &= \frac{1}{\varphi_u} C', & \bar{D} &= \frac{1}{\varphi_u\psi_v} D, \end{aligned}$$

the invariants $H, K, w^{(u)}, w^{(v)}$ being cogredient with D . Equations (18) are invariant under this transformation, and it there-

fore follows that our canonical form is undisturbed thereby. The most general transformation of the group

$$x = \lambda \bar{x}, \quad \bar{u} = \varphi(u), \quad \bar{v} = \psi(v)$$

which leaves our canonical form invariant is obtained by placing $\lambda = \text{constant}$.

The quadratic form

$$(21) \quad (R/L)(L \, du^2 + N \, dv^2)$$

is absolutely invariant under the transformations (14) and (19) and vanishes for the asymptotic net. The cubic form

$$(22) \quad (R/L)(L \, \mathcal{C}' du^3 - 3L \, \mathcal{B}' du^2 \, dv - 3N \, \mathcal{C}' du \, dv^2 + N \, \mathcal{B}' dv^3)$$

is absolutely invariant under the same transformations, and vanishes¹ for the curves of Darboux. By calculating the ratio of the discriminant² of (22) to the cube of the discriminant of (21) we find that this ratio is a constant. Hence the forms (21) and (22) are the forms Φ_2 and Φ_3 of Fubini. Therefore the choice of the proportionality factor which leads to our canonical form is that which gives Fubini's normal coordinates. The points x_u and x_v are therefore characterized geometrically as the points where the reciprocal of the projective normal of S_x crosses the parametric tangents.

The coefficients of Green's equations (12) in our canonical form, when calculated and expressed in terms of the invariants of N_x , may be written in the following form:

¹Lane, "Bundles and pencils of nets on a surface," Trans. Amer. Math. Soc., Vol. 28 (1926), p. 163.

²Fubini and Cech, Geometria proiettiva differenziale, Zanichelli, (1926), pp. 84-87.

$$\begin{aligned}
 \bar{a}' &= L/N, \\
 \bar{b}' &= -2\mathfrak{C}' - (\log L/N)_u/2 + (\log R)_u, \\
 \bar{c}' &= (L/N)(2\mathfrak{B}' + (\log L/N)_v/2 - (\log R)_v), \\
 \bar{d}' &= \mathfrak{D} - b\bar{b}' - a\bar{c}' + b_u - \bar{a}'a_v + b^2 - \bar{a}'a^2, \\
 \bar{a} &= \mathfrak{B}' + (\log R)_v/2, \\
 \bar{b} &= \mathfrak{C}' - (\log L/N)_u/2 + (\log R)_u/2, \\
 \bar{c} &= H - ab + a_u = K - ab + b_v.
 \end{aligned}
 \tag{23}$$

We shall now make use of Lane's results¹ in determining the projective normal of S_x at P_x . The point φ defined by

$$\varphi = x_{vv} + (N/L)(\bar{b} - 2\mathfrak{C}')x_u + (2\mathfrak{B}' - \bar{a} + (\log L/N)_v/2)x_v + \lambda x$$

is any point, except P_x , on the projective normal of S_x at P_x . By making use of the third equation of the canonical form of system (9) we may express φ as a linear combination of x , x_u , x_v , y . Upon equating to zero the coefficients of x_u and x_v we find the conditions that the line λ_{xy} coincides with the projective normal of S_x at P_x . We may state our result as follows:

The line λ_{xy} of the conjugate congruence coincides with the projective normal of S_x at P_x in case

$$\begin{aligned}
 (\bar{L}/\bar{N})\bar{\gamma} + \bar{\alpha} - (\log \bar{L}/\bar{N})_u/2 &= 0, \\
 (\bar{N}/\bar{L})\bar{\beta} + \bar{\delta} - (\log \bar{N}/\bar{L})_v/2 &= 0.
 \end{aligned}
 \tag{24}$$

We have now reached a canonical form of system (9) which is characterized analytically by the following conditions:

$$\begin{aligned}
 \alpha + b + A + (\log N)_u + 3(\log L/N)_u/2 - 2(\log R)_u &= 0, \\
 a + \delta + B + (\log L)_v + (\log L/N)_v/2 - 2(\log R)_v &= 0,
 \end{aligned}
 \tag{25}$$

¹Lane, "Contributions to the theory of conjugate nets," Amer. Journ. Math., Vol. 49, No. 4, (1927), p. 569.

$$(26) \quad \begin{aligned} (L/N)\gamma + \alpha - (\log L/N)_u/2 &= 0, \\ (N/L)\beta + \delta - (\log N/L)_v/2 &= 0. \end{aligned}$$

Conditions (25) imply that the line $x_u x_v$ is the reciprocal of the projective normal of S_x , and conditions (26) and (25) imply that the line λ_{xy} is the projective normal of S_x at P_x .

By making use of (25) and the seventh and eighth integrability conditions (10), we may write (26) in the simpler form

$$(27) \quad A = (\log R/L)_u, \quad B = (\log R/L)_v.$$

In order to treat S_x, S_y in a symmetrical manner we see that x, y satisfy a system of differential equations of the form (9) but with the roles of x and y interchanged. If the last two equations of (9) are solved for x_u and x_v , two of the desired equations result at once. The other three are found by differentiating the last two equations of (9) and eliminating the derivatives of x . We thus obtain the following equivalent system:

$$(28) \quad \begin{aligned} y_{uu} &= \bar{p}y + \bar{\alpha}y_u + \bar{\beta}y_v + \bar{L}x, \\ y_{uv} &= \bar{c}y + \bar{a}y_u + \bar{b}y_v, \\ y_{vv} &= \bar{q}y + \bar{\gamma}y_u + \bar{\delta}y_v + \bar{N}x, \\ x_u &= \bar{f}y + \bar{m}y_u + \bar{A}x, \\ x_v &= \bar{g}y + \bar{n}y_v + \bar{B}x, \end{aligned}$$

where the coefficients are indicated by dashes and given by the following¹ formulas:

$$(29) \quad \begin{aligned} \bar{p} &= A_u + mL - A(m_u/m + f/m + \alpha) - m\beta B/n, \\ \bar{\alpha} &= \alpha + f/m + m_u/m + A, \quad \bar{\beta} = m\beta/n, \\ \bar{L} &= -m(\alpha f/m + \beta g/n + (f/m)^2 - p - (f/m)_u), \end{aligned}$$

¹Lane, Projective Differential Geometry of Curves and Surfaces, p. 185.

$$\begin{aligned}
 \bar{c} &= A_v - A(m_v/m + a) - B(f/n + mb/n), \\
 &= B_u - B(n_u/n + b) - A(g/m + na/m), \\
 \bar{a} &= a + m_v/m = B + g/m + na/m, \\
 \bar{b} &= b + n_u/n = A + f/n + mb/n, \\
 (29) \quad \bar{q} &= B_v + nN - B(n_v/n + g/n + \delta) - n\gamma A/m, \\
 \bar{\gamma} &= n\gamma/m, \quad \bar{\delta} = \delta + g/n + n_v/n + B, \\
 \bar{N} &= -n(\delta g/n + \gamma f/m + (g/n)^2 - q - (g/n)_v), \\
 \bar{f} &= -A/m, \quad \bar{m} = 1/m, \quad \bar{A} = -f/m, \\
 \bar{g} &= -B/n, \quad \bar{n} = 1/n, \quad \bar{B} = -g/n.
 \end{aligned}$$

In order that S_y may be non-developable we shall suppose that $\bar{L} \bar{N} \neq 0$.

In view of the fact that it is possible to treat the two surfaces S_x, S_y in a symmetrical way, we shall assume that the coefficients of system (23), which are defined in (29), satisfy conditions analogous to (25) and (26). We thus have

$$\begin{aligned}
 (30) \quad \bar{\alpha} + \bar{b} + \bar{A} + (\log \bar{N})_u + 3(\log \bar{L}/\bar{N})_u/2 - 2(\log \bar{R})_u &= 0, \\
 \bar{a} + \bar{\delta} + \bar{B} + (\log \bar{L})_v + (\log \bar{L}/\bar{N})_v/2 - 2(\log \bar{R})_v &= 0,
 \end{aligned}$$

and

$$\begin{aligned}
 (31) \quad (\bar{L}/\bar{N})\bar{\gamma} + \bar{\alpha} - (\log \bar{L}/\bar{N})_u/2 &= 0, \\
 (\bar{N}/\bar{L})\bar{\beta} + \bar{\delta} - (\log \bar{N}/\bar{L})_v/2 &= 0.
 \end{aligned}$$

Conditions (30) imply that the line $y_u y_v$ is the reciprocal of the projective normal of S_y at P_y , and conditions (30) and (31) imply that the line χ_{xy} is the projective normal of S_y at P_y . Thus we arrive at the following theorem:

The conditions that we have imposed insure that every pair of integral surfaces of system (9) will be in relation F, the conjugate congruence being the projective normal congruence and the parametric conjugate nets on both surfaces being the projective lines of curvature.

IV. HARMONIC CONGRUENCE

The tangent planes at corresponding points of the two surfaces S_x, S_y intersect in the lines of the harmonic congruence of the transformation. In the Euclidean theory of parallel surfaces it is well known that the tangent planes at corresponding points are parallel. This means that all pairs of tangent planes intersect in lines that lie in the plane at infinity. Hence the developables of the harmonic congruence are indeterminate, since this congruence is a ruled plane. In seeking a projective analogue of this metric property we shall assume that the developables of the harmonic congruence are indeterminate, that is, that corresponding tangent planes of the two surfaces S_x, S_y intersect in the lines of a fixed plane. After obtaining considerable analytic simplification of our results by means of this assumption, we conclude this section by formulating our definition of projective parallelism and a theorem describing the analytic apparatus for studying it.

The line of intersection of the tangent planes at two corresponding points P_x, P_y of the surfaces S_x, S_y joins the points P_p, P_σ defined by

$$(32) \quad \rho = x_u + fx/m, \quad \sigma = x_v + gx/n,$$

as is seen by inspecting the last two equations of (9). When u, v vary, this line generates a congruence $\rho\sigma$. By the usual method¹ we may find the differential equation of the developables of the congruence $\rho\sigma$ to be

¹Lane, Projective Differential Geometry of Curves and Surfaces, pp. 185-186.

$$(33) \quad (m\bar{L}\bar{N} - nN\bar{L})du dv = 0 .$$

A necessary and sufficient condition that the developables of the congruence $\rho\sigma$ be indeterminate is seen from (33) to be

$$(34) \quad \bar{L}/\bar{N} = mL/nN .$$

We shall suppose from now on that this condition is satisfied.

Conditions (30) and (31) may now be simplified. If the expressions for $\bar{\alpha}$ and $\bar{\delta}$ found in (29) be substituted in (31) and use be made of (26) and (34), we may write (31) in the form

$$(35) \quad \begin{aligned} f/m + A + (\log mn)_u/2 &= 0, \\ g/n + B + (\log mn)_v/2 &= 0. \end{aligned}$$

Finally, by making use of (27), we may write these conditions

$$(36) \quad \begin{aligned} f/m &= - (\log \sqrt{mn} R/L)_u, \\ g/n &= - (\log \sqrt{mn} R/L)_v. \end{aligned}$$

It is evident from the integrability conditions that (25) and (30) may be written respectively in the following forms:

$$(37) \quad \begin{aligned} \alpha + 2b - L\gamma/N + (\log (L/N)^{3/2} \cdot (1/R^2))_u &= 0, \\ \delta + 2a - N\beta/L + (\log (L/N)^{1/2} \cdot (1/R^2))_v &= 0, \end{aligned}$$

and

$$(38) \quad \begin{aligned} \bar{\alpha} + 2\bar{b} - \bar{L}\bar{\gamma}/\bar{N} + (\log (\bar{L}/\bar{N})^{3/2} \cdot (1/\bar{R}^2))_u &= 0, \\ \bar{\delta} + 2\bar{a} - \bar{N}\bar{\beta}/\bar{L} + (\log (\bar{L}/\bar{N})^{1/2} \cdot (1/\bar{R}^2))_v &= 0. \end{aligned}$$

If we replace $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\delta}$, $\bar{a}$, $\bar{b}$ in (38) by their expressions given in (29) and make use of (34), (35), and (37), we may write (38) in the form

$$(\log mR/\bar{R})_u = 0, \quad (\log mR/\bar{R})_v = 0,$$

from which it is immediately seen that

$$(39) \quad \bar{R} = c m R \quad (c = \text{const.}).$$

By use of (29) and (34), two of the invariants for the net N_y , indicated by dashes, corresponding to those for N_x in (13), are found to have the following expressions:

$$(40) \quad \bar{\mathcal{B}}' = \mathcal{B}' + (\log m)_v/2, \quad \bar{\mathcal{C}}' = \mathcal{C}' + (\log n)_v/2.$$

The most general transformation of the form

$$x = h\bar{x}, \quad y = k\bar{y}$$

which leaves the form of the canonical system (9) invariant has h and k constant. The only coefficients not absolutely invariant under such a transformation are those for which we find

$$\begin{aligned} \bar{L} &= kL/h, & \bar{N} &= kN/h, \\ \bar{f} &= hf/k, & \bar{g} &= hg/k, \\ \bar{m} &= hm/k, & \bar{n} &= hn/k. \end{aligned}$$

It is therefore possible by a suitable choice of h and k to make the constant appearing in (39) equal to unity. We shall therefore have

$$(41) \quad \bar{R} = mR.$$

Now the expression for $\bar{R}$ corresponding to that for R given in (13) may be calculated easily by making use of (34) and (40). If this expression, together with the expression for R given in (13), is substituted in (41), we find after some simplification the following result:

$$(42) \quad mL(1-n)\mathcal{B}'^2 + nN(1-m)\mathcal{C}'^2 + m_vL(\mathcal{B}' + m_v/4n) +$$

$$(42) \quad + n_u N(\mathfrak{S}' + n_u/4n) = 0.$$

We may now state the following summary:

If the developables of the harmonic congruence are indeterminate, condition (42) implies that the line $y_u y_v$ is the reciprocal of the projective normal of S_y at P_y , while conditions (42) and (36) imply that the line χ_{xy} is the projective normal of S_y at P_y .

At this point we shall deduce two results of which later use will be made. It is evident that (27) is invariant under the transformations (29). Hence by use of (29) and (41) we may write equations (27) in the form

$$f/m = - (\log mR/\bar{R})_u, \quad g/n = - (\log mR/\bar{R})_v.$$

Finally, by making use of (36) we find

$$(43) \quad \bar{L} = c \sqrt{m/n} L, \quad \bar{N} = c \sqrt{n/m} N,$$

where the constant c appearing in both equations must be the same constant in order that (34) may be satisfied.

We are now in position to formulate our definition of projectively parallel surfaces in the following words:

Two surfaces S_x, S_y are said to be projectively parallel in case they are in the relation of a fundamental transformation with the projective normal congruence as the conjugate congruence and with the developables of the harmonic congruence indeterminate.

Our results may be described analytically in the following theorem:

Equations (9) with coefficients which satisfy conditions (25), (27), (36) and (42) define a pair of projectively parallel surfaces S_x, S_y in ordinary projective space S_3 , except for a

projective transformation.

We conclude this section with the remark that the projective analogue of the plane at infinity will be studied in Section VII.

V. SURFACES PARALLEL TO A GIVEN SURFACE

In the Introduction we remarked that in the metric sense there are infinitely many surfaces parallel to a given surface. In this section we investigate the analogous property with reference to the configuration which we have constructed, and seek to find out whether a given surface admits of infinitely many projectively parallel surfaces.

If we make the transformation

$$(44) \quad y = \lambda x + \mu z \quad (\lambda, \mu \text{ scalars})$$

on system (9), subject to the conditions which we have thus far imposed, we find that x, z are solutions of differential equations of the form

$$(45) \quad \begin{aligned} x_{uu} &= \tilde{p}x + \tilde{\alpha}x_u + \tilde{\beta}x_v + \tilde{L}z, \\ x_{uv} &= \tilde{c}x + \tilde{a}x_u + \tilde{b}x_v, \\ x_{vv} &= \tilde{q}x + \tilde{\eta}x_u + \tilde{\delta}x_v + \tilde{N}z, \\ z_u &= \tilde{f}x + \tilde{m}x_u + \tilde{A}z, \\ z_v &= \tilde{g}x + \tilde{n}x_v + \tilde{B}z, \end{aligned}$$

wherein the coefficients are indicated by curls. The only coefficients needed at the moment are those given by the expressions

$$(46) \quad \begin{aligned} \tilde{f} &= (f + \lambda A - \lambda_u)/\mu, & \tilde{g} &= (g + \lambda B - \lambda_v)/\mu, \\ \tilde{m} &= (m - \lambda)/\mu, & \tilde{n} &= (n - \lambda)/\mu. \end{aligned}$$

After an inspection of the last two equations of (45) it is evident that the tangents to the curves of the net N_z through P_z intersect the corresponding tangents to the curves of the net N_x through P_x .

Let us determine the condition that the tangent plane to S_z at P_z contains the line $\rho\sigma$ defined by (32). In order that this condition may be satisfied we must have

$$\begin{aligned}x_u + fx/m &= x_u + \tilde{f}x/\tilde{m}, \\x_v + gx/n &= x_v + \tilde{g}x/\tilde{n},\end{aligned}$$

whence by (46)

$$(47) \quad f/m = (f + \lambda A - \lambda_u)/(m - \lambda), \quad g/n = (g + \lambda B - \lambda_v)/(n - \lambda).$$

By simple reduction and use of (35) we find

$$(48) \quad (\log \lambda)_u = -(\log \sqrt{mn})_u, \quad (\log \lambda)_v = -(\log \sqrt{mn})_v,$$

from which it is evident that

$$(49) \quad \lambda = c/\sqrt{mn} \quad (c = \text{const.}).$$

With the aid of (47) the seventeen coefficients of (45) may be written as follows:

$$\begin{aligned}(50) \quad \tilde{p} &= p + \lambda L, & \tilde{\alpha} &= \alpha, & \tilde{\beta} &= \beta, & \tilde{L} &= \mu L, \\ \tilde{c} &= c, & \tilde{a} &= a, & \tilde{b} &= b, \\ \tilde{q} &= q + \lambda N, & \tilde{\gamma} &= \gamma, & \tilde{\delta} &= \delta, & \tilde{N} &= \mu N, \\ \tilde{f} &= f(m - \lambda)/\mu m, & \tilde{m} &= (m - \lambda)/\mu, & \tilde{A} &= A - \mu_u/\mu, \\ \tilde{g} &= g(n - \lambda)/\mu n, & \tilde{n} &= (n - \lambda)/\mu, & \tilde{B} &= B - \mu_v/\mu.\end{aligned}$$

With the definition of λ in (49), these coefficients satisfy all of the integrability conditions (10) of system (45).

By the use of (29) we are able to calculate the coefficients of the differential equations of the form (45) but with x and z interchanged. The only coefficients needed are those given by the formulas

$$(51) \quad \bar{L} = ((m - \lambda)/\mu)(\bar{L}/m + \lambda_L), \quad \bar{N} = ((n - \lambda)/\mu)(\bar{N}/n + \lambda_N).$$

By an inspection of (13) and (50) we may easily verify the following relations:

$$(52) \quad \tilde{B}' = B', \quad \tilde{C}' = C', \quad \tilde{R} = R.$$

We are now in position to investigate certain conditions which must be satisfied by the coefficients of (45) if the surface S_z is to be projectively parallel to S_y . It is readily seen that the coefficients defined by (50) satisfy (8), from which we conclude that the parametric net on S_z is the conjugate net N_z .

Conditions (25) take the following form when use is made of (27):

$$\alpha + b = (\log R \sqrt{N/L})_u, \quad a + \delta = (\log R \sqrt{N/L})_v.$$

It is easy to see from (50) and (52) that these conditions are satisfied by the coefficients of system (45). In a similar way conditions (27) are seen to be satisfied. If use be made of (51) and (50), after some simple reductions it may be verified without difficulty that condition (34) is likewise satisfied.

We note that μ introduced in (44) has been entirely arbitrary thus far, and its determination has been unnecessary in our calculations. However, this is not the case when we seek to determine whether conditions (36) and (42) are satisfied by the coefficients (50) of system (45). A value of μ which, with proper choices made from (50) and (52), will satisfy (36) is given by

$$(53) \quad \mu = \sqrt[4]{(m - \lambda)(n - \lambda)/mn}.$$

It is now possible to show by means of (50), (52), and (53) that condition (42) is not satisfied identically by the coefficients

of system (45). We therefore arrive at an additional condition which must be imposed on our configuration. This condition may be written in the form

$$\begin{aligned}
 (54) \quad & (m - \lambda L(1 - (n - \lambda)/\mu) \mathcal{B}'^2 + (n - \lambda N(1 - (m - \lambda)/\mu) \mathcal{C}'^2 \\
 & + \mu L((m - \lambda)/\mu)_v [\mathcal{B}' + (\log (m - \lambda)/\mu))_v/4] \\
 & + \mu N((n - \lambda)/\mu)_u [\mathcal{C}' + (\log (n - \lambda)/\mu))_u/4] = 0.
 \end{aligned}$$

We may now summarize our results as follows:

Two projectively parallel surfaces S_x, S_y do not in general admit of infinitely many projectively parallel surfaces S_z , without the imposition of a further restriction on the configuration.

VI. MODIFIED PROJECTIVE PARALLELISM

In the preceding section, contrary to the facts of the situation in the metric case, it was found that a given surface did not in general admit of infinitely many projectively parallel surfaces. We shall now study in the familiar way the configuration composed of two surfaces S_x, S_y in ordinary space S_3 , in the relation of a fundamental transformation, dropping, however, in this section alone, the hypothesis that the common conjugate congruence is the projective normal congruence. This configuration will be studied by means of a completely integrable system of partial differential equations, which Lane¹ has shown can be reduced to the form

$$\begin{aligned}
 x_{uu} &= px + \alpha x_u + \beta x_v + Ly, \\
 x_{uv} &= ax_u + bx_v, \\
 x_{vv} &= qx + \gamma x_u + \delta x_v + Ny, \\
 y_u &= mx_u, \quad y_v = nx_v \quad (mnLN \neq 0).
 \end{aligned}
 \tag{55}$$

In fact, this is merely our system (9) in which $c = f = g = A = B = 0$. The coefficients of this system satisfy the following integrability conditions:

$$\begin{aligned}
 a_u + ab &= \alpha_v + \beta\gamma, & b_v + ab &= \delta_u + \beta\gamma, \\
 b_u + b^2 + a\beta &= \beta_v + b\alpha + nL + \beta\delta + p, \\
 a_v + a^2 + b\gamma &= \gamma_u + a\delta + mN + \alpha\gamma + q, \\
 p_v &= ap - q\beta, & q_u &= bq - p\gamma, \\
 L_v &= aL - \beta N, & N_u &= bN - \gamma L,
 \end{aligned}
 \tag{56}$$

¹Lane, Projective Differential Geometry of Curves and Surfaces, pp. 172-173.

$$(56) \quad m_v = a(n - m), \quad n_u = b(m - n).$$

The most general transformation of the form

$$(57) \quad x = \lambda \bar{x}, \quad y = \mu \bar{y},$$

which leaves the form of system (55) invariant, has λ and μ constant. The only coefficients not absolutely invariant under such a transformation are m, n, L, N , for which we find

$$(58) \quad \bar{m} = \lambda m / \mu, \quad \bar{n} = \lambda n / \mu, \quad \bar{L} = \mu L / \lambda, \quad \bar{N} = \mu N / \lambda.$$

The transformation

$$(59) \quad \bar{u} = \varphi(u), \quad \bar{v} = \psi(v)$$

changes system (55) into another system whose coefficients are given by

$$(60) \quad \begin{aligned} \bar{p} &= p / \varphi_u^2, & \bar{\alpha} &= (\alpha - \varphi_{uu} / \varphi) / \varphi_u, & \bar{\beta} &= \psi_v \beta / \varphi_u^2, \\ \bar{L} &= L / \varphi_u^2, & \bar{a} &= a / \psi_v, & \bar{m} &= m \end{aligned}$$

and the formulas obtainable from these by the substitution

$$(61) \quad \begin{pmatrix} u & a & m & \alpha & p & \gamma & L \\ v & b & n & \delta & q & \beta & N \end{pmatrix}.$$

The coefficients of the equations corresponding to (55) when the roles of x and y are interchanged are indicated by dashes and are given by the following expressions:

$$(62) \quad \begin{aligned} \bar{p} &= mL, & \bar{\alpha} &= \alpha + m_u / m, & \bar{\beta} &= m \beta / n, & \bar{L} &= mp, \\ \bar{a} &= na / m, & \bar{b} &= mb / n, & \bar{m} &= l / m, & \bar{n} &= l / n, \\ \bar{q} &= nN, & \bar{\gamma} &= n \gamma / m, & \bar{\delta} &= \delta + n_v / n, & \bar{N} &= nq. \end{aligned}$$

We shall suppose that $pq \neq 0$ in order that S_y may be non-developable.

The line of intersection of the tangent planes at two corresponding points P_x, P_y of the surfaces S_x, S_y joins the points P_ρ, P_σ defined by

$$(63) \quad \rho = x_u, \quad \sigma = x_v.$$

The condition that the developables of the congruence of lines $\rho\sigma$ shall be indeterminate is

$$(64) \quad \overline{L/N} = mL/nN.$$

By use of (62) this condition may be written

$$(65) \quad p/q = L/N.$$

It is not difficult to show by means of this condition and the integrability conditions that

$$L_v/L = p_v/p, \quad N_u/N = q_u/q.$$

Therefore we have

$$L/p = U(u), \quad N/q = V(v),$$

and by a transformation of parameters we may make U and V each equal to unity. Hence

$$(66) \quad p = L, \quad q = N.$$

System (55) may now be written in the form

$$(67) \quad \begin{aligned} x_{uu} &= L(x + y) + \alpha x_u + \beta x_v, \\ x_{uv} &= ax_u + bx_v, \\ x_{vv} &= N(x + y) + \gamma x_u + \delta x_v, \\ y_u &= mx_u, \quad y_v = nx_v, \quad (mnLN \neq 0). \end{aligned}$$

The integrability conditions for system (67) are found to be

$$\begin{aligned}
 (68) \quad & a_u + ab = \alpha_v + \beta\gamma, \quad b_v + ab = \delta_u + \beta\gamma, \\
 & b_u + b^2 + a\beta = \beta_v + b\alpha + nL + \beta\delta + p, \\
 & a_v + a^2 + b\gamma = \gamma_u + a\delta + mN + \alpha\gamma + q, \\
 & L_v = aL - \beta N, \quad N_u = bN - \gamma L, \\
 & m_v = a(n - m), \quad n_u = b(m - n).
 \end{aligned}$$

We may therefore describe our results in the following theorem:

Equations (67), with the integrability conditions (68) satisfied, define a pair of surfaces S_x, S_y , except for a projective transformation, which are projectively parallel in the modified sense.

Let us choose a point P_z on the line λ_{xy} joining corresponding points P_x, P_y of the two surfaces S_x, S_y such that

$$(69) \quad y = (1 + k)z + kx \quad (k = \text{const.}).$$

If we make the transformation (69) on system (67) we find that x, z are solutions of differential equations of the form

$$\begin{aligned}
 (70) \quad & x_{uu} = \tilde{L}(x + z) + \tilde{\alpha}x_u + \tilde{\beta}x_v, \\
 & x_{uv} = \tilde{a}x_u + \tilde{b}x_v, \\
 & x_{vv} = \tilde{N}(x + z) + \tilde{\gamma}x_u + \tilde{\delta}x_v, \\
 & z_u = \tilde{m}x_u, \quad z_v = \tilde{n}x_v
 \end{aligned}$$

wherein the coefficients are indicated by curls and defined by the formulas

$$\begin{aligned}
 (71) \quad & \tilde{p} = \tilde{L} = (1 + k)L, \quad \tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta, \\
 & \tilde{a} = a, \quad \tilde{b} = b, \quad \tilde{m} = (m - k)/(1 + k), \quad \tilde{n} = (n - k)/(1 + k), \\
 & \tilde{q} = \tilde{N} = (1 + k)N, \quad \tilde{\gamma} = \gamma, \quad \tilde{\delta} = \delta.
 \end{aligned}$$

The integrability conditions (68) are satisfied identically by these coefficients.

If we differentiate the fourth equation of (70) with respect to v and make use of the second, fourth, and fifth of equations (70) we find

$$z_{uv} = (a + m_v/(m - k))z_u + b((m - k)/(n - k))z_v.$$

Since this is a Laplace equation the parametric net N_z is conjugate.

When use is made of (62) and (71) we find

$$\tilde{L} = (m - k)L, \quad \tilde{N} = (n - k)N.$$

Using these expressions and the expressions for $\tilde{m}$, $\tilde{n}$, $\tilde{L}$, $\tilde{N}$ found in (71), we see that condition (64) is satisfied. Therefore the tangent planes at corresponding points of the two surfaces S_x , S_z intersect in the lines of a fixed plane. Furthermore, it is easy to show that the tangent planes at corresponding points P_x , P_y , P_z of the three surfaces S_x , S_y , S_z intersect in the same line by differentiating equation (69) to obtain

$$\begin{aligned} y_u &= (1 + k)z_u + kx_u, \\ y_v &= (1 + k)z_v + kx_v. \end{aligned}$$

In order that the three surfaces S_x , S_y , S_z may be distinct we shall exclude the cases for which $k = 0$ and $k = -1$.

We now reach the following conclusion, which we state in the form of a theorem:

Two surfaces S_x , S_y , projectively parallel in the modified sense, admit of infinitely many surfaces S_z projectively parallel to them in this modified sense.

VII. PROPERTIES OF PROJECTIVE PARALLELISM

We shall now return to the case where the conjugate congruence of the transformation F is the congruence of projective normals. The purpose of this section is to study further the projective differential geometry of our configuration. Certain projective properties which are analogues of metric properties of parallel surfaces are determined. A projective analogue is found for the so-called plane at infinity property of metric parallelism of surfaces, namely, that for a given surface and a parallel surface the cross ratio of the corresponding points of these two surfaces, the corresponding point of a parallel surface at a unit's distance from the given surface, and the point where the plane at infinity intersects the normal through the corresponding points, is constant. A problem which arises in the investigation of transformations F is to consider a conjugate net having a special property and to inquire under what conditions the F transformed net may have the same property. This problem is investigated for the nets N_x and N_y which are F transforms with the conjugate congruence of projective normals. A projective analogue of the metric relation between the points which are the Laplace transforms of points on a normal to a system of parallel surfaces and the focal points on this normal is determined. Employing a local coordinate system, we consider the axes and rays at corresponding points of projectively parallel surfaces. Finally, the correspondence between certain nets of curves on two projectively parallel surfaces is considered.

In metric geometry we recall not only that two parallel surfaces possess the same normal at corresponding surface points,

but that the distance measured along the normal between these points is constant, i.e., is the same for all pairs of corresponding points. We may describe this last property in a different way. Given a surface S_0 and a parallel surface S_h . If we denote by P_0 and P_h the two corresponding points of S_0 and S_h respectively, on the same normal, and if, on this normal, P_1 is a point on the parallel surface at a unit's distance from S_0 , and finally if P_∞ is the point where the plane at infinity cuts the normal, then

$$(P_\infty P_0 P_1 P_h) = h \quad (h = \text{const.}).$$

We shall now give an interesting projective analogue of the aforementioned metric property.

Green¹ has shown, and we have remarked earlier in this paper, that when the curves corresponding to the developables of a congruence of lines $\rho\sigma$ are indeterminate, the congruence consists entirely of lines lying in a fixed plane. Let us determine the point where our fixed plane through the line $\rho\sigma$ cuts the projective normal X_{xy} . This plane is determined by the two points given by (32) and by the point ρ_u (or σ_v). By differentiating with respect to u the expression for ρ found in (32), and making use of (9), (29) and (32), we may express ρ_u in the form

$$\rho_u = \bar{L}x/m + Ly + (\alpha + f/m)\rho + \beta\sigma.$$

We therefore see that the fixed plane intersects the projective normal X_{xy} in the point

$$(72) \quad \bar{L}x + mLy.$$

The same point may be written by symmetry, or by a similar compu-

¹Green, "Memoir on the general theory of surfaces, etc.," Trans. Amer. Math. Soc., Vol. 20, No. 2, p. 105.

tation using σ_v , in the form

$$\bar{N}x + nNy.$$

If use be made of (43) and (49) we may write (72) in the form

$$(73) \quad k\lambda x + y \quad (k = \text{const.}).$$

Let us now choose the point defined by (73) as the point corresponding to the point at infinity on the normal in the metric case, P_y corresponding to P_1 and P_z corresponding to P_h . We then find that

$$(74) \quad (P_y P_x P_z P) = (\infty, 0, \lambda, k\lambda) = k \quad (k = \text{const.}).$$

This result may be stated as follows:

On each projective normal common to three projectively parallel surfaces S_x, S_y, S_z , the cross ratio of the three corresponding points P_x, P_y, P_z and the point of intersection of this projective normal with the fixed plane containing the lines of the harmonic congruence is constant.

In case of modified projective parallelism discussed in Section VI it is easy to show that the fixed plane containing the lines $\rho\sigma$ of the harmonic congruence cuts the line λ_{xy} in the point

$$(75) \quad x + y.$$

Choosing this point as the point corresponding to the point at infinity on a normal in the metric case, we find in a similar way that

$$(P_y P_x P_z P_{x+y}) = (\infty, 0, k, 1) = 1/k = \text{const.}$$

Lane¹ has calculated certain invariants of a net and has computed the change in them due to the transformation F with a general conjugate congruence. In a similar way we may study the nets N_x , N_y which are F transforms with the special conjugate congruence of projective normals.

By use of (29), (34) and (35), some of the invariants for N_y , indicated by dashes, corresponding to those for N_x given by (13), are found to have the following expressions:

$$\begin{aligned}
 \bar{H} &= H - (\log m^3 n)_{uv}/2, & \bar{K} &= K - (\log mn^3)_{uv}/2, \\
 \bar{W}(u) &= W(u) + (\log mn^3)_{uv}/2, & \bar{W}(v) &= W(v) + (\log m^3 n)_{uv}/2, \\
 (76) \quad \bar{\mathcal{H}} &= \mathcal{H}, & \bar{\mathcal{K}} &= \mathcal{K}, \\
 \bar{\mathcal{B}}' &= \mathcal{B}' + (\log m)_v/2, & \bar{\mathcal{C}}' &= \mathcal{C}' + (\log n)_u/2.
 \end{aligned}$$

Thus, if N_x has certain projective properties, we can find from (76) the necessary and sufficient conditions that N_y has the same properties. For example, if N_x has equal point invariants, then N_y will also have equal point invariants if and only if $m/n = U(u)/V(v)$.² The ray curves then form conjugate nets on both S_x and S_y . If $H = 0$ the u -curves of N_x are cone curves; therefore, the u -curves are also cone curves of N_y if and only if $(\log m^3 n)_{uv} = 0$. Dually, the u -curves are plane curves of N_x if $\mathcal{H} = 0$, and in this case they are also plane curves of N_y . If N_x is isothermally conjugate, so that $W(u) = W(v)$, then N_y will also be characterized by this same property if and only if $m/n = U(u)/V(v)$. If N_x has equal tangential invariants, $\mathcal{H} = \mathcal{K}$, then N_y will also

¹Lane, "On the fundamental transformation of surfaces," Annals of Math., Vol. 30, No. 3, p. 462.

²Slotnick, "A contribution to the theory of fundamental transformations of surfaces," Trans. Amer. Math. Soc., Vol. 30 (1928), p. 190.

have equal tangential invariants, and the axis curves form conjugate nets on both S_x and S_y .

In metric geometry there is an interesting relation¹ between the points which are the Laplace transforms of points on the normal to a system of parallel surfaces and the focal points on the normal. The line joining the first (minus first) Laplace transforms intersects the normal in the first (second) focal point

Let us consider the projective analogue of this relation. The Laplace transformed points, or the ray-points of P_x with respect to N_x , are the points defined by

$$(77) \quad \sigma_x = x_v - ax, \quad \rho_x = x_u - bx.$$

The Laplace transformed points of N_y are given by

$$(78) \quad \sigma_y = y_v - \bar{a}y, \quad \rho_y = y_u - \bar{b}y.$$

The line joining σ_x and σ_y intersects the projective normal in the point

$$(g + na)x + (B - \bar{a})y.$$

This point coincides with the point defined by (5) if and only if

$$m(B - \bar{a}) = -g - na,$$

a condition which is seen to be satisfied by use of (29).

The Laplace transformed points of N_z are defined by

$$(79) \quad \sigma_z = z_v - \tilde{a}z, \quad \rho_z = z_u - \tilde{b}z,$$

wherein by use of (50) and (29) we have

¹Bompiani, "Corrispondenza fra una Superficie e le sue parallele," Mathematische Zeitschrift, Band 24, (1925-26), p. 312.

$$\bar{a} = a + (\log (m - \lambda)/\mu)_v, \quad \bar{b} = b + (\log (n - \lambda)/\mu)_u.$$

The line joining σ_x and σ_z intersects the projective normal in the point

$$[g + na - \lambda\{B + g/n - (\log (m - \lambda))_v\}]x + [B - a - (\log (m - \lambda))_v]y.$$

This point coincides with the point η if and only if

$$g + na - \lambda B - \lambda g/n + \lambda[\log (m - \lambda)]_v = -mB + ma + m[\log (m - \lambda)]_v.$$

By use of (10) and (47) this condition is easily shown to be satisfied. Hence the points σ_x , σ_y , σ_z lie on a straight line which passes through the focal point η on the projective normal. Similarly we may show that the points ρ_x , ρ_y , ρ_z lie on a line passing through the focal point ζ .

Any point X on the line joining the two Laplace transformed points ρ_x , ρ_y is defined by placing

$$(80) \quad X = \rho_x + k \rho_y \quad (k \text{ scalar}).$$

By use of the expressions for ρ_x and ρ_y found in (77) and (78), and the fourth of equations (9), we may express X as a linear combination of x , x_u , and y thus,

$$(81) \quad X = (kf - b)x + (1 + km)x_u - (k/n)(f + mb)y.$$

Equating to zero the coefficient of x_u in the expression thus obtained, we find the condition on k necessary and sufficient that the point X coincide with the focal point ζ on the projective normal, namely,

$$k = -1/m.$$

Moreover, eliminating x_u and y in (81) by means of (44) and (45),

we may express X as a linear combination of x , z , and z_u . Setting equal to zero the coefficient of x in this result and making use of (49) and (50), we find the condition on k necessary and sufficient that the point X coincide with the Laplace transformed point ρ_z to be

$$k = - (n/c) \sqrt{n/m} \quad (c = \text{const.}).$$

Therefore

$$(82) \quad (\rho_x, \rho_y, \rho_z, \zeta) = (0, \infty, - (n/c) \sqrt{n/m}, - 1/m) \\ = n\sqrt{mn}/c.$$

In a similar way we may show that

$$(83) \quad (\sigma_x, \sigma_y, \sigma_z, \eta) = (0, \infty, - (m/c) \sqrt{m/n}, - 1/n) = m\sqrt{mn}/c.$$

Our results may be stated as follows:

The cross ratio of the four points $\rho_x, \rho_y, \rho_z, \zeta$ is $n\sqrt{mn}/c$, and that of the points $\sigma_x, \sigma_y, \sigma_z, \eta$ is $m\sqrt{mn}/c$.

It is evident that these two point ranges are equicross only in case $m = n$.

If local coordinates $x_1, \dots, x_4$ based on the tetrahedron x, x_u, x_v, y with suitably chosen unit point are introduced, the ray of P_x with respect to N_x , which joins the Laplace transformed points ρ_x, σ_x defined by (77), is found to have the local equations

$$(84) \quad x_4 = x_1 + bx_2 + ax_3 = 0.$$

The ray of P_y with respect to N_y joins the points ρ_y, σ_y defined by (78), and the ray of P_z with respect to N_z joins the points ρ_z, σ_z defined by (79). A necessary and sufficient condition that the ray of P_x and the ray of P_y be coplanar is that in the

expressions for the four quantities $\rho_x, \sigma_x, \rho_y, \sigma_y$, the determinant of the coefficients of x, x_u, x_v, y vanish, i.e., that

$$(85) \quad \begin{vmatrix} -b & 1 & 0 & 0 \\ -a & 0 & 1 & 0 \\ f & m & 0 & (f+mb)/n \\ g & 0 & n & (g+na)/m \end{vmatrix} = 0.$$

On expanding this determinant, we obtain the equation

$$(86) \quad (m-n)(f+mb)(g+na) = 0.$$

In view of the property of collinearity of the corresponding ray-points of the nets N_x, N_y, N_z , the vanishing of the determinant (85) implies that the ray of P_z also lies in the plane determined by the ray of P_x and the ray of P_y . On the assumption that the ray points of P_x are distinct from the ray points of P_y and do not coincide with the points ρ, σ defined by (32), equation (86) shows that the three rays of corresponding points P_x, P_y, P_z of three projectively parallel surfaces S_x, S_y, S_z are coplanar if and only if $m = n$.

Any point Y on the ray of P_y can be defined by a linear combination of the form

$$(87) \quad Y = \rho_y + \tau \sigma_y \quad (\tau \text{ scalar}).$$

The point of intersection of the ray of P_y with the tangent plane of S_x at P_x may be found by determining τ so that in the expressions for the point Y and the points x, x_u, x_v , the determinant of the coefficients of x, x_u, x_v, y vanish. On expanding the determinant just referred to, we find that the ray of P_y intersects the tangent plane of S_x at P_x in the point Y for which

$$\tau = - \frac{m}{n} \frac{f + mb}{g + na} .$$

This point lies on the ray of P_x if and only if its coordinates satisfy the equations (84), a condition which is found without difficulty to be equivalent to $m = n$. In a similar way we may find the point where the ray of P_z meets the tangent plane of S_x at P_x . This point is found to lie on the ray of P_x , and likewise on the ray of P_y if $m = n$. Consequently, if the three rays of corresponding points P_x, P_y, P_z of three projectively parallel surfaces S_x, S_y, S_z are coplanar they are also concurrent.

It may be remarked that when $m = n$ the projective lines of curvature are indeterminate. Equations (5) show that the focal points on the projective normal coincide. By making use of the ninth and tenth of the integrability conditions (10) and equations (35), it may be shown that $m = n = \text{constant}$. Furthermore, the projective normal congruence consists of a bundle of lines with its center at the point $y - mx$. In this case the transformation F is a radial transformation.

The axis of P_x with respect to N_x is the line of intersection of the two osculating planes of the two curves C_u and C_v at P_x on S_x . The first equation of (12) may be written in the form

$$x_{uu} - (\alpha - L\gamma/N)x_u = Lx_{vv}/N + (\beta - L\delta/N)x_v + (p - Lq/N)x.$$

The left-hand member represents a point in the osculating plane to the curve C_u , and the right-hand member a point in the osculating plane to the curve C_v . Therefore the point P_ψ defined by

$$\psi = x_{uu} - (\alpha - L\gamma/N)x_u = Lx_{vv}/N + (\beta - L\delta/N)x_v + (p - Lq/N)x$$

represents a point on the line of intersection of the two

osculating planes. By making use of the first equation of (9) we have

$$(88) \quad \psi = px + L\gamma x_u/N + \beta x_v + Ly.$$

Consequently the local coordinates of the point P_ψ on the axis of P_x referred to the tetrahedron we are using are $(p, L\gamma/N, \beta, L)$. The local equation of the plane containing the axis of P_x and the projective normal of S_x at P_x is

$$(89) \quad \beta Nx_2 - \gamma Lx_3 = 0.$$

In a similar way a point on the axis of P_y , and the plane containing the axis of P_y and the projective normal of S_y at P_y may be found. This plane is identical to the plane (89); therefore, the axis of P_x and the axis of P_y are coplanar. It may be shown without difficulty that the axis of P_z , the corresponding point on a projectively parallel surface S_z , lies in this same plane.

The differential equation of the asymptotic curves on S_x is

$$(90) \quad L du^2 + N dv^2 = 0,$$

and the asymptotic curves on S_y are given by the equation

$$\bar{L} du^2 + \bar{N} dv^2 = 0,$$

which by use of (43) may be written

$$(91) \quad mL du^2 + nN dv^2 = 0.$$

It is evident by an inspection of (90) and (91) that the asymptotic curves on the surfaces S_x and S_y do not correspond when $m \neq n$.

This is the projective equivalent of the similar theorem in the metric theory of parallel surfaces.

